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EXOTIC ANALYTIC STRUCTURES AND EISENMAN INTRINSIC MEASURES

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ABSTRACT

Using Eisenman intrinsic measures we prove a cancellation theorem. This theorem allows to find new examples of exotic analytic structures on $\mathbf{C}^n$ under which we understand smooth complex affine algebraic varietiers which are diffeomorphic to $\mathbf{R}^{2n}$ but not biholomorphic to $\mathbf{C}^n$. We also develop a new method of constructing these structures which enables us to produce exotic analytic structures on $\mathbf{C}^3$ with a given number of hypersurfaces isomorphic to $\mathbf{C}^2$ and a family of these structures with a given number of moduli.

1. Introduction

A smooth complex affine algebraic variety is called an exotic analytic (resp. algebraic) structure on $\mathbf{C}^n$ if it is diffeomorphic to $\mathbf{R}^n$ but not biholomorphic (resp. isomorphic) to $\mathbf{C}^n$. The famous theorem of Ramanujam says that there are no exotic structures on $\mathbf{C}^2$ [R]. In his paper Ramanujam also constructed an example of a smooth complex contractible algebraic surface X which is not homeomorphic to $\mathbf{C}^2$. He mentioned that the direct product $X \times \mathbf{C}$ was diffeomorphic to $\mathbf{R}^6$ due to the h-cobordism theorem and asked if this product was biholomorphic to $\mathbf{C}^3$. Zaidenberg proved that the answer was negative [Z1], [Z2] and, thus, constructed the first example of exotic analytic structure on $\mathbf{C}^3$. His proof is based on the following analytic cancellation theorem: for every measure hyperbolic complex

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manifold Y of dimension k the direct product $Y \times \mathbf{C}^n$ cannot be biholomrphic to $\mathbf{C}^{n+k}$. The surface X in the example of Ramanujam has Kodaira logarithmic dimension 2 and, therefore, it is measure hyperbolic, by the Sakai theorem [Sa]. A weaker fact that $X \times \mathbf{C}$ is an exotic algebraic structure on $\mathbf{C}^3$ follows from the Iitaka–Fujita cancellation theorem proven earlier [IF].

In this paper we reprove the cancellation theorem of Zaidenberg using Eisenman intrinsic measures [E]. This theorem becomes a trivial corollary of the definition of Eisenman intrinsic measures and the answer to Ramanujam's question becomes a trivial corollary of Sakai's theorem. Moreover, these measures enable us to construct new examples of exotic analytic structures. In particular, we present an example of an exotic analytic structure M on $\mathbf{C}^3$ which contains a given number of surfaces that are biholomorphic to $\mathbf{C}^2$ (the author is grateful to M. Zaidenberg for the question about the existence of such examples). The reason for this question is the following nontrivial fact: every exotic analytic structure which is the direct product of a contractible algebraic surface of Kodaira logarithmic dimension 2 and a Euclidean space $\mathbf{C}^n$ does not contain an algebraic hypersurface biholomorphic to $\mathbf{C}^{n+1}$ [Z1]. Moreover we shall show that in this case there are no analytic hypersurfaces biholomorphic to $\mathbf{C}^{n+1}$. The exotic structure M mentioned above depends on a given number of parameters ("positions" of surfaces biholomorphic to $\mathbf{C}^2$). In other words this is a family of exotic analytic structures on $\mathbf{C}^3$ with a given number of moduli, since generic members of this family are not pairwise biholomorphic.

Examples of exotic algebraic structures on $\mathbf{C}^3$ with similar properties are known. For instance, an example of an exotic algebraic structure on $\mathbf{C}^3$ with a given number of moduli can be found in [FZ]. Another examples of exotic algebraic structures on $\mathbf{C}^3$ which contain a surface biholomorphic to $\mathbf{C}^2$ are presented in [PtD], [D], [Ka], [Ru]. But it is still unknown whether an exotic algebraic structure must be an exotic analytic structure. The only known examples of exotic analytic structures were constructed in [Z1] (it is also worth mentioning [W] where one can find an example of non-Stein algebraic variety which is diffeomorphic to $\mathbf{R}^8$).

2. Preliminaries

First introduce notation: M is a complex manifold of dimension n; TM is the holomorphic tangent bundle of M; T_pM is the holomorphic tangent space at

$p \in M$; $\Lambda^k T_pM$ (resp. Λ^k TM) is the k-th exterior power of T_pM (resp. TM); $D_p^k M$ is the set of decomposable elements in $\Lambda^k T_pM$. If $\langle , \rangle$ is a Hermitian metric on TM, then it can be extended to a Hermitian metric on Λ^k TM by putting

$$\langle \alpha, \beta \rangle \equiv \det\{\langle v_i, w_j \rangle\}$$

where $\alpha = v_1 \wedge \cdots \wedge v_k$ and $\beta = w_1 \wedge \cdots \wedge w_k$ from $D_p^k M$ and extending this definition by linearity to arbitrary elements of Λ^k TM. Let $\| \gamma \|$ be the Hermitian metric on $\Lambda^k TB^k$ (where B^k is the unit ball in $\mathbf{C}^k$) generated by the Bergman metric, and let o be the origin in B^k.

Definition 2.1: For every $p \in M$ and every $\alpha \in D_p^k M$ $(1 \leq k \leq n)$ the intrinsic Eisenman k-measure of α is $E_k^M(p, \alpha) \equiv \inf\{\| \gamma \|^2 \mid \gamma \in D_o^k B^k$ and there exists a holomorphic mapping $f\colon B^k \to M$ with $f(o) = p$ and $f_*(\gamma) = \alpha\}$.

This definition immediately implies the following theorem.

THEOREM 2.2 ([Ko]): *Let N be a complex manifold of dimension $\geq k$. Then for every holomorphic mapping $f\colon M \to N$ we have $f^* E_k^N(p, \alpha) \leq E_k^M(p, \alpha)$ for all $p \in M$ and $\alpha \in D_p^k M$.*

THEOREM 2.3 ([GW] Product Formula): *Let M and N be complex manifolds of dimensions n and n' respectively, and let $1 \leq k \leq n$, $1 \leq \ell \leq n'$. Let $p \in M$, $q \in N$, $\alpha \in D_p^k M$, $\beta \in D_q^\ell N$. Put $\gamma = \alpha \wedge \beta$. Then*

$$E_{k+\ell}^{M \times N}((p,q), \gamma) = E_k^M(p, \alpha) \cdot E_\ell^N(q, \beta).$$

Remark: We would also like to consider the case when k or l is 0. For $k = 0$ we shall treat the vector γ as the image of β under the mapping generated by the embedding $i_p\colon N \to M \times N$ given by $x \to (p, x)$ (similarly for $l = 0$). Put $E_0^M \equiv 1$, then Theorem 2.2 implies that the Product Formula also holds when k or ℓ is 0.

For $k = \dim M$ the Eisenman k-measure of M is called the Kobayashi–Eisenman pseudovolume. Recall that the manifold M is measure hyperbolic if its Kobayashi–Eisenman pseudovolume is not degenerate outside a subset of M of Hausdorff $2n$-measure 0. A smooth algebraic variety is of hyperbolic type if its dimension coincides with its Kodaira logarithmic dimension.

THEOREM 2.4 ([Sa]): *If a smooth algebraic variety is of hyperbolic type, then it is measure hyperbolic.*

It is known (e.g., see [PtD]) that every smooth contractible complex algebraic surface X which is not $\mathbf{C}^2$ has Kodaira logarithmic dimension either 1 or 2. In the second case X is measure hyperbolic, by Theorem 2.4.

THEOREM 2.5 ([Z1]): *Let X be a smooth contractible complex algebraic surface of hyperbolic type (i.e. its Kodaira logarithmic dimension is 2). Then for every natural k there are no injective regular mappings $\mathbf{C}^k \longrightarrow X \times \mathbf{C}^{k-1}$.*

Remark: As Zaidenberg mentions [Z1] a stronger fact holds for $k = 1$. Namely there is no proper holomorphic injection $\mathbf{C} \longrightarrow X$. Indeed, every such injection can be extended to a holomorphic mapping of the Riemann sphere to a smooth completion of X [NS, Theorem 7]. By the Chow theorem this extended mapping is regular, but this contradicts Theorem 2.5.

We shall also need the following well-known corollary of the h-cobordism theorem (e.g., see [Ka]).

THEOREM 2.6: *Every smooth contractible affine algebraic variety of dimension $n \geq 3$ is diffeomorphic to $\mathbf{R}^{2n}$.*

3. Main lemmas

Theorems 2.2, 2.3 and 2.6 immediately imply Zaidenberg's cancellation theorem in a stronger form.

LEMMA 3.1: *Let $M_1, ..., M_s$ be contractible smooth complex algebraic varieties. Suppose that $\dim M_i = n_i$, $l \geq 0$, $m = \sum_{i=1}^s n_i + l$ and the Eisenman intrinsic k_i-measure is not trivial on M_i, where $k_i \geq 0$ (for $k_i = 0$ see the remark after Theorem 2.3). Let $k = k_1 + \cdots + k_s > 0$ and $m \geq 3$. Then $M = M_1 \times \cdots \times M_s \times \mathbf{C}^l$ is an exotic analytic structure on $\mathbf{C}^m$. Moreover, if the Eisenman intrinsic measure $E_{k_i}^{M_i}$ does not vanish identically at some point $p_i \in M_i$ for every $i = 1, ..., s$ then E_k^M does not vanish identically at $p = p_1 \times \cdots \times p_s \times z$ for every $z \in \mathbf{C}^l$*

Proof: The statement about the Eisenman intrinsic measure E_k^M follows from Theorem 2.3. Since for $k > 0$ the Eisenman intrinsic k-measure of $\mathbf{C}^m$ is trivial,

M cannot be biholomorphic to $\mathbf{C}^m$, by Theorem 2.2. On the other hand it is diffeomorphic to $\mathbf{R}^{2m}$, by Theorem 2.6. ∎

The proof of the following proposition is mostly a repetition of some arguments of Fujita [F]. Nevertheless the fact should be formulated.

PROPOSITION 3.2: *Let M be a smooth complex algebraic variety whose first and second cohomology groups are trivial, and let $\overline{M}$ be its smooth completion for which the divisor $D = \overline{M} - M$ is of simple normal crossing type. Consider the algebra $A(M)$ of regular functions on M, the multiplicative group $A(M)^*$ of its invertible elements, the Neron–Severi group $NS(\overline{M})$ and the Picard group $\mathrm{Pic}(\overline{M})$ of $\overline{M}$, and the free Abelian group $G(D)$ with basis consisting of the prime components of D. Then*

(i) $A(M)^* = \mathbf{C}^*$;

(ii) *$A(M)$ is a unique factorization domain;*

(iii) *the Picard group $\mathrm{Pic}(M)$ of M is trivial;*

(iv) *the groups $\mathrm{Pic}(\overline{M}), NS(\overline{M})$ and $G(D)$ are isomorphic.*

Proof: By the universal coefficient formula, $H_1(M) = 0$. Clearly, the embedding $M \to \overline{M}$ generates a surjective homomorphism of the first homology groups. Therefore, $H_1(\overline{M}) = 0$. Again, by the universal coefficients formula, $H^1(\overline{M}) = 0$ and $H^1(\overline{M}, \mathbf{C}) = 0$. By Hodge theory, $H^{0,1}(\overline{M}) \oplus H^{1,0}(\overline{M}) = H^1(\overline{M}, \mathbf{C})$ and, therefore, $H^{0,1}(\overline{M}) = 0$. This implies that the Picard manifold $\mathrm{Pic}_0(\overline{M}) = H^{0,1}(\overline{M})/H^1(\overline{M})$ is also trivial. Let $NS(M)$ be the Neron–Severi group of M. Then we have the following exact sequences [F, (1.18)]

(a) $0 \to A(M)^*/\mathbf{C}^* \to G(D) \to \mathrm{Pic}(\overline{M}) \to \mathrm{Pic}(M) \to 0$,

(b) $0 \to \tilde{H}^1(M) \to G(D) \to NS(\overline{M}) \to NS(M) \to 0$,

(c) $0 \to NS(M) \to \hat{H}^2(M)$,

(d) $0 \to A(M)^*/\mathbf{C}^* \to \tilde{H}^1(M) \to \mathrm{Pic}_0(\overline{M}) \to \mathrm{Pic}_0(M) \to 0$,

where $\hat{H}^q(M)$ (resp. $\tilde{H}^q(M)$) is a subgroup (resp. a quotient group) of $H^q(M)$. Since the groups $\hat{H}^q(M)$, $\tilde{H}^q(M)$ (for $q = 1$ and 2) and the group $\mathrm{Pic}_0(\overline{M})$ are trivial, one can see that $A(M)^* \cong \mathbf{C}^*$ and $\mathrm{Pic}_0(M) = 0$, by (d); $NS(M) = 0$ by (c); $G(D) \cong NS(\overline{M})$ by (b). Since $NS(\overline{M}) = \mathrm{Pic}(\overline{M})/\mathrm{Pic}_0(\overline{M})$, we have $G(D) \cong \mathrm{Pic}(\overline{M})$ and (a) implies that $\mathrm{Pic}(M) = 0$. Then [F, (1.20)] implies that $A(M)$ is a unique factorization domain. ∎

LEMMA 3.3: *Let M be a smooth complex affine algebraic variety such that $H^1(M) = H^2(M) = 0$ and let F be a hypersurface in M. Suppose that a smooth closed subvariety C of M is contained in the smooth part F^* of F and that $\operatorname{codim}_M C \geq 2$. Consider the monoidal transformation $\tau\colon \tilde{M} \to M$ with locus at C. Let $\tilde{F}$ be the proper transform of F. Then the manifold $N = \tilde{M} - \tilde{F}$ is affine.*

Proof: Let $\overline{M}$ be a smooth completion of M for which the divisor $D = \overline{M} - M$ is of simple normal crossing type. Denote by $\overline{F}$ the closure of F in $\overline{M}$. Let $G(D)$ be the free Abelian group with basis consisting of the prime components of D and let $p\colon G(D) \to \operatorname{Pic}(\overline{M})$ be the natural embedding. By Proposition 3.2, $\operatorname{Pic}(M) = 0$. Since $\operatorname{Pic}(M) = \operatorname{Pic}(\overline{M})/p(G(D))$, this means that p is surjective. In particular, there exists a meromorphic function f on $\overline{M}$ so that $(f) - \overline{F} \in G(D)$, where (f) is the divisor of f. Thus the restriction of f to M is regular, $F = \{x \in M|\ f(x) = 0\}$, and f has simple zeros on F^*. Let $f_1,\ldots,f_s$ be generators in the ideal of regular functions on M that vanish on C (suppose that f is one of them), and let $x_1,\ldots,x_m$ be a coordinate system of $\mathbf{C}^m$ which contains M as a closed affine subvariety. Put $g_i = \left(\frac{f_i}{f}\right) \circ \nu$ for $i = 1,\ldots,s$ where ν is the restriction of τ to N. Our aim is to show that these functions are regular on N and that the mapping $\Phi = (x_1 \circ \nu, \ldots, x_m \circ \nu, g_1, \ldots, g_s)\colon N \to \mathbf{C}^{m+s}$ is a proper embedding, i.e. N is affine. Let $\tilde{E}$ be the exceptional divisor of the monoidal transformation τ and let $E = \tilde{E} - \tilde{F}$. By construction, $f_i \circ \nu$ has zeros on E and $f \circ \nu$ has simple zeros on E. Therefore g_i is regular on N. Consider a small neighborhood $U \subset M$ of a point $c_0 \in C$. One may suppose that in this neighborhood the germs of the submanifolds F and C are given in a local coordinate system $(y_1, \ldots, y_n)$ by $y_{l+1} = 0$ and $y_1 = \cdots = y_{l+1} = 0$ respectively. Put $V = \nu^{-1}(U)$ and $z_i = (y_i/y_{l+1}) \circ \nu$. Recall that $\tau|_{\tilde{E}}\colon \tilde{E} \to C$ is a fibration with generic fiber $\mathbf{CP}^\ell$, where $\ell = n - \dim C - 1$. The intersection of $\tilde{F}$ and each of these fibers is isomorphic to $\mathbf{CP}^{\ell-1}$ and, thus, $\nu|_E\colon E \to C$ is a fibration with generic fiber $\mathbf{C}^\ell$. Locally this means that for every $c \in U \cap C$ the fiber $\tau^{-1}(c)$ admits the homogeneous coordinate system $y_1\colon \cdots\colon y_{l+1}$ in which $\tilde{F} \cap \tau^{-1}(c)$ is given by $y_{l+1} = 0$. Hence the functions z_i $(i = 1, \ldots l)$ are regular on V and can be viewed as coordinates for each fiber $\nu^{-1}(c) \cong \mathbf{C}^l$ where $c \in U \cap C$. Hence the mapping $(y_1 \circ \nu, \ldots, y_n \circ \nu, z_1, \ldots, z_l)\colon V \to U \times \mathbf{C}^l$ is a proper embedding. Let J be the ideal of holomorphic functions on U generated by $y_1, \ldots, y_{l+1}$. Using linear combinations, one may suppose that $f_i - y_i \in J^2$

for $i = 1, \ldots, l$ and that $f = f_{l+1} = y_{l+1}h$ where h is invertible holomorphic on U. This means that in our small neighborhood U one can replace the coordinate system $(y_1, \ldots, y_n)$ by the coordinate system $(f_1, \ldots, f_{l+1}, y_{l+2} \ldots, y_n)$, for it has the same properties. In this last system $g_i = z_i$ in V for $i = 1, \ldots, l$. Hence $(\hat{y}_1 \circ \nu, \ldots, \hat{y}_n \circ \nu, g_1, \ldots, g_s)$: $V \to U \times \mathbf{C}^s$ is a proper embedding for every coordinate system $(\hat{y}_1, \ldots, \hat{y}_n)$ in U. Since the functions $x_1, \ldots, x_m$ distinguish the points of C the mapping Φ is a proper embedding as well. ∎

LEMMA 3.4: *Let M be a complex manifold and let F be an irreducible hypersurface in M. Suppose that a closed submanifold C of M is contained in the smooth part F^* of F and $\operatorname{codim}_M C \geq 2$. Consider the monoidal transformation τ: $\tilde{M} \to M$ with locus at C. Let $\tilde{F}$ be the proper transform of F and let $N = \tilde{M} - \tilde{F}$. Then the fundamental groups of N and M are isomorphic.*

Proof: Consider the natural embeddings i: $M - F \to M$, j: $M - F \to N$, and the projection $\nu = \tau|_N$: $N \to M$. These mappings generate homomorphisms i_*, j_*, and ν_* of the fundamental groups $\pi_1(M - F), \pi_1(M)$, and $\pi_1(N)$. Clearly, i_*, j_*, and ν_* are surjective. We have to show that ν_* is an isomorphism, i.e. $\operatorname{Ker} \nu_* = 0$. Consider a small disc $d \subset M$ which meets F normally at a point $c_0 \in C$. Suppose that a simple loop γ in $d - c_0$ generates a subgroup $\Gamma \subset \pi_1(M - F)$. Let Γ_n be the smallest normal subgroup of $\pi_1(M - F)$ that contains Γ. Since F is irreducible, $\operatorname{Ker} i_* = \Gamma_n$. Let $\tilde{d}$ be the proper transform of d and $\tilde{\gamma} = \tau^{-1}(\gamma)$. Note that $\tilde{d}$ does not meet $\tilde{F}$ and, in particular, $\tilde{d} \subset N$. Hence $\tilde{\gamma}$ is contractible in N and $\operatorname{Ker} j_* \supset \Gamma_n$. Since $i_* = \nu_* \circ j_*$ we have $\operatorname{Ker} j_* = \Gamma_n$ and $\operatorname{Ker} \nu_* = 0$. ∎

THEOREM 3.5: *Let M be a smooth complex contractible affine algebraic variety of dimension ≥ 2. Let F be an irreducible hypersurface in M such that either F is smooth contractible or F is homeomorphic to a Euclidean space. Suppose that a smooth closed contractible subvariety C of M is contained in the smooth part of F and $\operatorname{codim}_M C \geq 2$. Consider the monoidal transformation τ: $\tilde{M} \to M$ with locus at C. Let $\tilde{F}$ be the proper transform of F. Then the manifold $N = \tilde{M} - \tilde{F}$ is affine and contractible. In particular, N is diffeomorphic to a Euclidean space when $\dim M \geq 3$, by Theorem 2.6.*

Proof: The algebraic manifold N is affine and has a trivial fundamental group, by Lemmas 3.3 and 3.4. We have to show now that the senior homology groups of N are also trivial. Let $\tilde{E}$ be the exceptional divisor of the monoidal transformation τ, let $E = \tilde{E} - \tilde{F}$, and let $\mu = \tau|_E$. Recall that μ: $E \to C$ is a fibration with generic fiber $\mathbf{C}^\ell$. The exact homotopy sequence of this fibration and the Whitehead theorem imply that the hypersurface $E \subset N$ is contractible. Let $\dot{M}$ be a one-point compactification of M and let $\dot{N}$ be a one-point compactification of N. Denote the closure of E in $\dot{N}$ by $\dot{E}$ and the closure of F in $\dot{M}$ by $\dot{F}$. Note that $\dot{E}$ and $\dot{F}$ are homology $(2n-2)$-spheres and $\dot{M}$ is a homology $2n$-sphere. Indeed, consider, for instance, $\dot{E}$. For $n \neq 3$ this is true since E is diffeomorphic to $(2n-2)$-dimensional real Euclidean space, by Theorem 2.6 (E is affine since N is affine). Let $n = 3$ then E is a contractible surface. Consider its smooth completion $\hat{E}$ and the curve $D = \hat{E} - E$. By a theorem of Ramanujam [R], D is connected, $\pi_1(\hat{E}) = \pi_1(D) = 0$, and the embedding $D \hookrightarrow \hat{E}$ generates an isomorphism of the second homology groups. By Poincaré duality, the third homology of $\hat{E}$ is trivial. Taking into consideration the fact that $\hat{E}/D = \dot{E}$ one can see from the exact homology sequence of the pair $(\hat{E}, D)$ that the homology of $\dot{E}$ coincides with the homology of a 4-dimensional sphere.

Since $\dot{M}$ is a homology $2n$-sphere, the exact cohomology sequence of the pair $(\dot{M}, \dot{F})$ implies $H^{2n-1}(\dot{M}/\dot{F}) = H^{2n}(\dot{M}/\dot{F}) = \mathbf{Z}$ and all the other senior cohomology groups of $\dot{M}/\dot{F}$ are trivial. Note that $M - F = \dot{M} - \dot{F}$ is homeomorphic to $N - E = \dot{N} - \dot{E}$ and, therefore, $\dot{M}/\dot{F}$ and $\dot{N}/\dot{E}$ are homeomorphic. Then the exact cohomology sequence of the pair $(\dot{N}, \dot{E})$ implies that $H^{2n}(\dot{N}) = \mathbf{Z}, H^k(\dot{N}) = 0$ for $0 < k \leq 2n-3$, and the exact sequence

$$0 \to H^{2n-2}(\dot{N}) \to \mathbf{Z} \to \mathbf{Z} \to H^{2n-1}(\dot{N}) \to 0$$

holds. Put $p = \dot{N} - N$. By Poincaré duality, $H_{2n-q}(N) = H^q_{\text{comp}}(\dot{N} - p) = H^q(\dot{N}, p) = \dot{H}^q(\dot{N})$, where $\dot{H}^q(\dot{N})$ is the reduced cohomology group. Hence the senior homology groups of N, except for possibly the first and second groups, are trivial. But the first group is trivial since the fundamental group of N is trivial. Thus we have the exact sequence

$$0 \to H_2(N) \to \mathbf{Z} \to \mathbf{Z} \to 0$$

from which follows that $H_2(N)$ is also trivial. Hence N is contractible, by the Whitehead theorem. ∎

Remark: The assumption that $C \subset F^*$ is not essential in Lemma 3.3, but it makes the proof easier. It is essential in Lemma 3.4, as the following example shows. Let $M = \mathbf{C}^2$, $F = \{x^2 - y^3 = 0\}$ (where (x, y) is a coordinate system in M), and let C be the origin. Suppose that N is constructed from this data in the same way it was done in Lemma 3.4. Then one can check that the fundamental group of N is non-trivial. The procedure described in Theorem 3.5 is similar a "half point attachment" in [F]. When $M = \mathbf{C}^2$ this procedure may generate contractible surfaces of Kodaira logarithmic dimension 1. For instance, consider again $F = \{x^2 - y^3 = 0\}$, but choose C to be a regular point on F. Then N is contractible of Kodaira logarithmic dimension 1 [PtD].

Recall that a complex manifold Y is Liouville if there is no bounded non-constant holomorphic functions on Y.

LEMMA 3.6: *Let X be a smooth contractible algebraic surface of Kodaira logarithmic dimension 2 and let Y be a Liouville simply connected manifold of dimension $\geq k$ whose Eisenman intrinsic 2-measure is trivial at each point. Suppose that N is a complex manifold of dimension $k-1$, and suppose that $f: Y \longrightarrow X \times N$ is a proper holomorphic mapping. Let $b \in N$ and let $R = f^{-1}(X \times b)$ be non-empty. Then the restriction of f to R cannot be an injection.*

Proof: Let p_X: $X \times N \longrightarrow X$ be the natural projection and $g = p_X \circ f$. If $\dim g(Y) = 2$ then one may suppose that g has rank 2 at some point $q \in Y$ and the Kobayashi–Eisenman pseudovolume of X is non-trivial at $g(q)$. By Theorem 2.2, the Eisenman intrinsic 2-measure of Y is non-trivial at q which contradicts to the assumption of lemma. Thus $\Lambda = g(Y)$ does not contain a two-dimensional complex manifold. Put $G = f(R)$ and assume that $f|_R$: $R \to X \times N$ is an injection. Since f is proper, G is a closed analytic subset of $X \times b$. The mapping $p_X|_G$: $G \to \Lambda$ is an embedding and thus $\dim R = \dim G \leq 1$. Since the codimension of $X \times b$ in $X \times N$ is $k - 1$ the codimension of R in Y is $\leq k - 1$, i.e. $\dim R \geq 1$. Hence R and G are curves. In particular, Λ contains a closed curve $p_X(G)$. Since Y is a connected manifold, Λ is irreducible. This implies that $\Lambda = p_X(G)$, i.e. the mapping $p_X|_G : G \longrightarrow \Lambda$ is biholomorphic. Every loop on R is contractible in Y, since Y is simply connected by the assumption of lemma. Hence the image of this loop in $\Lambda = g(Y) = p_X(G)$ is contractible, i.e. Λ and, therefore, G and R are simply connected. Hence the normalization Q of Λ must be biholomorphic either to a disc or to $\mathbf{C}$. Suppose that Q is a disc, then every bounded non-constant holomorphic function on Q generates a bounded non-constant continuous function φ on Λ which is holomorphic on $\Lambda - S$ where

S is the set of the singular points of Λ. Note that $\varphi \circ g$ is a bounded continuous nonconstant function on Y which is holomorphic on $Y - g^{-1}(S)$. Since $g^{-1}(S)$ is a hypersurface, $\varphi \circ g$ is actually holomorphic on Y. Hence the case of a disc must be cancelled, since Y is Liouville. The normalization Q of Λ cannot be biholomorphic to $\mathbf{C}$ as well since otherwise we have a proper holomorphic injection $Q \to \Lambda \subset X$ and this contradicts the remark after Theorem 2.5. Therefore the assumption that $f|_R$ is an injection is wrong. ∎

Remark: The original formulation of this lemma was weaker. This formulation appeared in discussion with M. Zaidenberg.

COROLLARY 3.7: *Let X be a smooth complex contractible algebraic surface of Kodaira logarithmic dimension 2. Then for every natural k there is no proper holomorphic injection $\mathbf{C}^{k+1} \longrightarrow X \times \mathbf{C}^k$.*

4. Examples

(a) In the case when all M_i's are of hyperbolic type, Lemma 3.1 is one of the statements of the analytic cancellation theorem of Zaidenberg [Z1], otherwise this lemma presents other exotic analytic structures. For example, we may suppose that $s = 2$, M_1 is of hyperbolic type and M_2 is a contractible algebraic surface of Kodaira logarithmic dimension 1.

Problem: The author does not know if Lemma 3.1 works in the case when all M_i's are smooth contractible algebraic surfaces of Kodaira logarithmic dimension 1. We can reformulate this problem as follows. Is it true that every smooth contractible algebraic surface of Kodaira logarithmic dimension 1 has either a non-trivial Kobayashi–Eisenman pseudovolume or a nontrivial Kobayashi–Royden pseudometric?

(b) Let X be a smooth contractible algebraic surface of hyperbolic type. Put $M = X \times \mathbf{C}$, $F = X \times 0$, a let C be a point $z \in F$. Then M is an exotic analytic structure by Zaidenberg's theorem. The triple (M, F, C) satisfies the assumptions of Theorem 3.5. Let N and τ be the same as in Theorem 3.5, then N is diffeomorphic to $\mathbf{R}^6$. Consider a generic point $p \in X$. The Kobayashi–Eisenman pseudovolume of X is non-trivial at p by Sakai's theorem. The Eisenman intrinsic 2-measure of M is non-trivial at point $q = (p, 1)$, by Theorem 2.2. Since the mapping $\tau|_N \colon N \to M$ is dominant the Eisenman intrinsic measure of N is non-trivial at $\tau^{-1}(q)$, again by Theorem 2.2. Hence N is an exotic analytic structure on $\mathbf{C}^3$. Note that this exotic structure contains a closed surface $\tau^{-1}(z) \cap N$ which is isomorphic to $\mathbf{C}^2$.

The construction of an exotic analytic structure on $\mathbf{C}^3$ with a given number of surfaces biholomorphic to $\mathbf{C}^2$ is quite similar. Consider points $a_1, ..., a_n \in \mathbf{C}$ and put $F(a_k) = X \times a_k$. Choose points $z_k \in F(a_k)$, and consider the blow-up $\pi \colon \tilde{M} \longrightarrow M$ at the points $z_1, ..., z_n$. Let $\tilde{F}(a_k)$ be the proper transform of $F(a_k)$. Then $\tilde{M} - \bigcup_{k=1}^{n} \tilde{F}(a_k)$ is diffeomorphic to $\mathbf{R}^6$, by Theorem 3.5. Same argument as above shows that $\tilde{M} - \bigcup_{k=1}^{n} \tilde{F}(a_k)$ has a non-trivial Eisenman intrinsic 2-measure and thus it is an exotic analytic structure on $\mathbf{C}^3$. We denote this exotic structure by $P(A, Z)$ in order to emphasize that it depends on the parameters $A = (a_1, ..., a_n) \in \mathbf{C}^n$ and $Z = (z_1, ..., z_n) \in X^n$. Let $E^k = \pi^{-1}(F(a_k)) - \tilde{F}(a_k)$. Then each E^k is isomorphic to $\mathbf{C}^2$.

LEMMA 4.1: *Except for $E^1, \ldots, E^n$ there are no closed surfaces in $P(A, Z)$ that are biholomorphic to $\mathbf{C}^2$.*

Proof: Assume that there is a proper holomorphic embedding $f \colon \mathbf{C}^2 \longrightarrow P(A, Z)$ for which $f(\mathbf{C}^2)$ is not contained in $\bigcup_{k=1}^{n} E^k$. Consider $g = \pi \circ f \colon \mathbf{C}^2 \longrightarrow M$. The restriction of g to $\mathbf{C}^2 - g^{-1}(\bigcup_{k=1}^{n} F_k)$ is an embedding. But this contradicts Lemma 3.6 and our assumption is wrong. ∎

LEMMA 4.2: *Let $(A, Z) = (a_1, \ldots, a_n, z_1, \ldots, z_n)$ and $(A', Z') = (a'_1, \ldots, a'_n, z'_1, \ldots, z'_n)$ be generic points in the space of parameters $\mathbf{C}^n \times X^n$. Then the exotic analytic structures $P(A, Z)$ and $P(A', Z')$ are not biholomorphic.*

Proof: Note that $P(A, Z) - \bigcup_{k=1}^{n} E^k$ is biholomorphic to $X \times (\mathbf{C} - A)$. Suppose that $\Phi \colon P(A, Z) \longrightarrow P(A', Z')$ is a biholomorphism. Since Φ maps every surface biholomorphic to $\mathbf{C}^2$ onto a surface biholomorphic to $\mathbf{C}^2$, then, by Lemma 4.1, we obtain a biholomorphism $\Psi \colon X \times (\mathbf{C} - A) \longrightarrow X \times (\mathbf{C} - A')$. Let $p \colon X \times (\mathbf{C} - A') \longrightarrow \mathbf{C} - A'$ be the natural projection, let $i_z \colon \mathbf{C} - A \longrightarrow X \times (\mathbf{C} - A)$ be the embedding given by $w \longrightarrow (z, w)$ where $z \in X$ and $w \in \mathbf{C} - A$, and let $\psi_z = p \circ \Psi \circ i_z \colon \mathbf{C} - A \longrightarrow \mathbf{C} - A'$. Consider the case $n \geq 2$. Suppose that $\psi_z \neq \text{const}$ for some z. Since every non-constant mapping $\mathbf{C} - A \to \mathbf{C} - A'$ is biholomorphic and the number of these mappings is finite we see that $\psi_z = \psi$ does not depend on z and it can be viewed as a linear fractional transformation of $\mathbf{C}$ that maps A onto A'. When $n > 2$ and A and A' are generic there is no linear fractional transformation of $\mathbf{C}$ with this property.

Suppose that $n = 2$. If ψ_z does not depend on z, then $\Psi(z,w) = (\varphi(z,w), \psi(w))$ where $z \in X, w \in \mathbf{C} - A$ and for every w the mapping $z \to \varphi(z,w)$ is a biholomorphism $X \to X$. Since the number of these biholomorphic mappings is finite [T], we see that $\varphi(z,w) = \varphi(z)$ does not depend on w. One may suppose that $\psi = id$, then the equality $\varphi(Z) = Z'$ must hold, i.e. in this case Z and Z' are not in a general position.

It remains to show that the functions ψ_z cannot be constant for every $z \in X$. Let $j_w \colon X \longrightarrow X \times (\mathbf{C} - A)$ be the embedding given by $z \longrightarrow (z, w)$ where $z \in X$ and $w \in \mathbf{C} - A$. Consider $\chi_w = p \circ \Psi \circ j_w \colon X \longrightarrow \mathbf{C} - A'$. Since the disc Δ is the universal covering of $\mathbf{C} - A'$ when $n \geq 2$ and X is contractible, χ_w generates a mapping from X to Δ which must be constant. Hence χ_w is constant, in other words the image of $j_w(X)$ under the mapping $p \circ \psi$ is constant. If for every z $\psi_z = \text{const}$, i.e. the image of every set $i_z(\mathbf{C} - A)$ under the mapping $p \circ \Psi$ is constant then the image of $X \times (\mathbf{C} - A)$ under this mapping is also constant. It is not true since Ψ is biholomorphic.

When $n = 1$ one may suppose that $A = A' = 0$ and Ψ is a biholomorphic mapping of $X \times \mathbf{C}^*$ onto itself. Theorem 1.10 from [Z1] implies that Ψ has the form $\Psi(z,w) = (\varphi(z), \psi(z,w))$ where $(z,w) \in X \times \mathbf{C}^*$ and φ is a biregular automorphism of X. Hence $\varphi(Z) = Z'$. Now the statement of lemma follows from the fact that the number of automorphisms of X is finite [T]. ∎

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